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Note

The Use of the Schur-Cohn Conditions for Determining the Stability Condition of Multi-Level Finite Difference Schemes

Kusic and Lavi in a paper in this journal [2] put forward a proposition that the Schur–Cohn conditions [3] might be used to imply the von Neumann necessary conditions for stability in the L_2 norm. Their proposition is the following:

PROPOSITION 1. *If the following conditions are met for the difference approximation to Eq.* (1) (*the linear constant coefficient partial differential equation of Kusi and Lavi's paper* [2]):

(a) *Definite difference schemes* (*as defined in Ref.* [2]) *are used to approximate the space derivatives*;

(b) *The lowest Δt-term of each Schur–Cohn determinant of the characteristic polynomial is of the proper algebraic sign and is nonzero except at*

$$k = 0, 2\pi/\Delta x, 4\pi/\Delta x, \ldots,$$

then the difference approximation satisfies the von Neumann necessary condition for stability as $\Delta t \to 0$ for a fixed Δx.

As it reads, Proposition 1 is meaningless. Stability should be examined for all values of the mesh ratios (unconditional stability) or at least some range of the mesh ratios (conditional stability) and to suggest Δx be held fixed while Δt tends to zero implies that all mesh ratios tend to zero and so no range of stability is obtained whatsoever. In the limit when $\Delta t = 0$ all difference schemes will degenerate into $\mathbf{V}^{n+1} = I\mathbf{V}^n$, where $\mathbf{V}^n$ is a vector denoting the solution to the difference scheme at $t = n\Delta t$. Clearly, then in this limit with Δx remaining fixed consistency is not obtained and so neither is convergence. It is, however, true that if Δt is chosen to be sufficiently small then higher order terms in Δt can be ignored. Nevertheless, Δt sufficiently small implies the mesh ratios sufficiently small and so only conditional stability is obtained. A verification of this statement is demonstrated by the following example:

Example. Consider the differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

and a consistent finite-difference approximation

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} = \frac{1}{(\Delta x)^2} [(1 - \theta)\, \delta_x^2 U_i^n + \theta \delta_x^2 U_i^{n+1}], \tag{α}$$

where

$$0 \leqslant \theta \leqslant 1.$$

Fourier transformation of the space variables yields

$$\mu - 1 = - \frac{4 \sin^2 \frac{k\Delta x}{2}}{(\Delta x)^2} [(1 - \theta) + \theta\mu](\Delta t),$$

where μ is the amplification factor; and so

$$\left[1 + \theta \frac{4 \sin^2 \frac{k\Delta x}{2}}{(\Delta x)^2} \cdot \Delta t\right] \mu - \left[1 - (1 - \theta) \frac{4 \sin^2 \frac{k\Delta x}{2}}{(\Delta x)^2} \cdot \Delta t\right] = 0.$$

Therefore

$$\begin{aligned} \Delta_1 &= \begin{vmatrix} a_0 & a_1 \\ \bar{a}_1 & \bar{a}_0 \end{vmatrix} = a_0^2 - a_1^2 \\ &= \left[1 - (1 - \theta) \frac{4 \sin^2 \frac{k\Delta x}{2}}{(\Delta x)^2} \cdot \Delta t\right]^2 \left[1 + \theta \frac{4 \sin^2 \frac{k\Delta x}{2}}{(\Delta x)^2} \cdot \Delta t\right]^2 \\ &= - \frac{8 \sin^2 \frac{k\Delta x}{2}}{(\Delta x)^2} \cdot \Delta t + \frac{16 \sin^4 \frac{k\Delta x}{2}}{(\Delta x)^4} (1 - 2\theta)(\Delta t)^2. \end{aligned}$$

Since von Neumann's condition is both necessary and sufficient for two level schemes [4] if Δt is chosen to be sufficiently small, the term in $(\Delta t)^2$ can be neglected, and we have a sufficient condition for the stability of (α). However, it is far from necessary as Δt can take any value we please when $\theta > \frac{1}{2}$ and stability is still maintained. In fact if we wish to find the correct stability range of $r = (\Delta t)/(\Delta x)^2$ (expressed as a function of θ) *it is imperative* that we include all the terms of Δ_1, in the analysis.

Finally, the idea of using the Schur–Cohn conditions as a test for stability is not entirely new. Douglas and Gunn [1] used the Hurwitz condition to show that the real parts of the roots of a real, conformally transformed, polynomial were negative. The Hurwitz criterion applied to this polynomial is equivalent to the Schur–Cohn conditions for showing that the roots of the original real polynomial lie in the unit circle.

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